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AUGMENTED MODAL DIMENSIONS FOR SEISMIC EVALUATION OF BUILDINGS

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Abstract

An N-degree-of-freedom structure can be represented by using N single-degree-of-freedom modal sticks/oscillators, which form an N-dimensional modal space. Because each modal stick represents a modal equation of motion, we always take it for granted that a modal stick is one-dimensional. This paper shows that a modal equation of motion could be multi-dimensional. Accordingly, the corresponding modal stick/oscillator is also multi-dimensional. In other words, we can augment the dimensions of a modal space. It is worth recalling that with the imaginary unit, the one-dimensional real number axis was augmented into the two-dimensional complex plane, which is influential in mathematics and science. Likewise, augmented modal dimensions are influential in seismic evaluation of structures. For instance, a three-degree-of-freedom modal stick can simultaneously reflects the three force-displacement relationships of a two-way asymmetrical building subjected to its modal inertia force. Moreover, a multi-degree-of-freedom modal stick allows the modal responses in the three directions to be different from each other. This feature is appealing for modal analysis of non-proportionally damped buildings, such as buildings with supplemental damping, or buildings with soil-structure interaction. On the other hand, it is common to use multiple tuned mass dampers to control a translation-rotation coupled vibration mode. Nevertheless, based on the concept of augmented modal dimensions, a translation-rotation coupled tuned mass damper was developed to control a translation-rotation coupled vibration mode. That novel tuned mass damper is a straightforward and simple solution to the modal control of asymmetrical buildings. Moreover, through conducting modal fusion, a self-mass damper, designated as top-story mass damper, was developed for the seismic control of the first triplet of vibration modes of two-way asymmetrical buildings. In short, augmented dimensions of modal space push the boundaries of seismic evaluation of structures.

Keywords: modal dimension, modal space, modal analysis, structural dynamics, seismic analysis and design, asymmetric-plan building, non-proportionally damped building, tuned mass damper.

1. Introduction

In physical space, the degree of freedom of a numerical model of a building can be very large or small. The smallest dimension is of course equal to one. An N-degree-of-freedom structure can be represented by using N single degree-of-freedom (SDOF) modal sticks, which form an N-dimensional modal space. Because each SDOF modal stick reflects a modal equation of motion, it always takes for granted that a modal stick is a one-dimensional oscillator. The SDOF modal oscillator is a cornerstone of structural dynamics and earthquake engineering. Nevertheless, Lin and Tsai [1, 2] demonstrated that a modal equation of motion can be expressed in a format of multiple degrees of freedom. Accordingly, the corresponding modal stick is no longer a one-dimensional oscillator. In other words, the dimension of the original N-dimensional modal space is augmented. The augmented modal dimension is useful for many aspects of earthquake engineering. It is worth recalling that augmenting the one-dimensional real axis into a two-dimensional complex plane through complex numbers significantly advanced mathematics, physics, and engineering.

Asymmetrical buildings are vulnerable to earthquakes because of their translation-rotation coupled response. It is also challenging to do seismic evaluation and design of such buildings. For a two-way asymmetrical building, every vibration mode simultaneously has two translational and one rotational deformation, i.e., a three-dimensional deformation. Therefore, using one-dimensional modal systems for asymmetrical buildings is inconsistent with the visual reality. Moreover, translational and

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rotational modal properties are mixed up (i.e., unclear) in terms of SDOF modal systems. Finally, SDOF modal systems can not reflect the fact that the modal deformations in the three directions are possibly non-proportional to each other, e.g., buildings with supplemental damping. In contrast, a three-degree-of-freedom (3DOF) modal equation of motion explicitly reveals those properties of a vibration mode of a two-way asymmetrical building [1, 2]. It is like to discover the ratios between protons, neutrons, and electrons in composition of an atom mass. Accordingly, some trends in torsional effects in asymmetrical buildings were further understood [3]. Moreover, for non-proportionally damped two-way asymmetrical buildings, the 3DOF modal equation of motion maintains the characteristic of non-proportional damping at a modal level. That characteristic benefits the modal response history analysis of asymmetrical buildings with supplemental damping [4, 5, 6]. Because the 3DOF modal system can reflect the non-proportionality between modal translations and modal rotation while asymmetrical buildings go into inelastic excursions, the augmented modal dimensions also benefit the seismic analysis of inelastic asymmetrical buildings [1, 2]. Based on the encouraging results, Lin et al. [7] further applied the augmented modal dimensions to the seismic analysis of buildings with specific vertical irregularities.

Moreover, Lin et al. [8, 9] applied the augmented modal dimensions to develop novel tuned mass dampers (TMDs). In contrast with the typical approach of employing multiple SDOF TMDs for the seismic control of asymmetrical buildings, the proposed approach used a single translation-rotation coupled TMD straightforwardly controlling a translation-rotation coupled vibration mode. Through modal fusion of the first triplet of vibration modes, which are the fundamental vibration modes in each of the three directions, Lin [10, 11] further developed a novel self-mass damper that simultaneously controls the three fundamental vibration modes.

The present paper shows the derivation of the 3DOF modal equation of motion as well as numerical examples. The present paper aims at briefing the theory and application of the augmented modal dimensions for seismic evaluation of buildings.

2. Theoretical background

2.1. 3DOF modal equation of motion

In order to be consistent with the coordinate system used in previous study [1, 2], the X- and Z-axes are used as the two plan axes. The positive direction of the Y-axis is opposite to the direction of gravity. The center of rigidity (CR) of a two-way asymmetric-plan buildings is not aligned with the center of mass (CM) in both plan axes. The two horizontal components of seismic ground motions are simultaneously applied along the x- and the z-directions. The equation of motion for an elastic N-story two-way asymmetric-plan building with each floor simulated as a rigid diaphragm is:

$$\begin{aligned}\mathbf{M\ddot{u}}+\mathbf{C\dot{u}}+\mathbf{Ku}&=-\mathbf{M}\boldsymbol{\iota}_x\ddot{u}_{gx}(t)-\mathbf{M}\boldsymbol{\iota}_z\ddot{u}_{gz}(t)\\&=-\sum_{n=1}^{3N}\mathbf{s}_{xn}\ddot{u}_{gx}(t)-\sum_{n=1}^{3N}\mathbf{s}_{zn}\ddot{u}_{gz}(t)=-\sum_{n=1}^{3N}\Gamma_{xn}\mathbf{M}\boldsymbol{\varphi}_n\ddot{u}_{gx}(t)-\sum_{n=1}^{3N}\Gamma_{zn}\mathbf{M}\boldsymbol{\varphi}_n\ddot{u}_{gz}(t)\\&=-\sum_{n=1}^{3N}\left(\Gamma_{xn}\ddot{u}_{gx}+\Gamma_{zn}\ddot{u}_{gz}\right)\mathbf{M}\boldsymbol{\varphi}_n\end{aligned}\tag{1}$$

where the displacement vector, $\mathbf{u}$, mode shape, $\boldsymbol{\varphi}_n$, mass matrix, $\mathbf{M}$, stiffness matrix, $\mathbf{K}$, influence vectors, $\boldsymbol{\iota}_x$, $\boldsymbol{\iota}_z$, modal inertia force vectors, $\mathbf{s}_{xn}$, $\mathbf{s}_{zn}$, and modal participation factors, Γ_{xn}, Γ_{zn}, are defined as:

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$$\mathbf{u}=\begin{bmatrix}\mathbf{u}_x\\\mathbf{u}_z\\\mathbf{u}_\theta\end{bmatrix}_{3N\times1},\quad \boldsymbol{\varphi}_n=\begin{bmatrix}\boldsymbol{\varphi}_{xn}\\\boldsymbol{\varphi}_{zn}\\\boldsymbol{\varphi}_{\theta n}\end{bmatrix}_{3N\times1},\quad \mathbf{M}=\begin{bmatrix}\mathbf{m}_x&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{m}_z&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{I}_0\end{bmatrix}_{3N\times3N},\quad \mathbf{K}=\begin{bmatrix}\mathbf{k}_{xx}&\mathbf{k}_{xz}&\mathbf{k}_{x\theta}\\\mathbf{k}_{zx}&\mathbf{k}_{zz}&\mathbf{k}_{z\theta}\\\mathbf{k}_{\theta x}&\mathbf{k}_{\theta z}&\mathbf{k}_{\theta\theta}\end{bmatrix}_{3N\times3N}$$

$$\boldsymbol{\iota}_x=\begin{bmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{0}\end{bmatrix},\quad \boldsymbol{\iota}_z=\begin{bmatrix}\mathbf{0}\\\mathbf{1}\\\mathbf{0}\end{bmatrix},\quad \mathbf{s}_{xn}=\Gamma_{xn}\mathbf{M}\boldsymbol{\varphi}_n,\quad \mathbf{s}_{zn}=\Gamma_{zn}\mathbf{M}\boldsymbol{\varphi}_n,\quad \Gamma_{xn}=\frac{\boldsymbol{\varphi}_n^T\mathbf{M}\boldsymbol{\iota}_x}{\boldsymbol{\varphi}_n^T\mathbf{M}\boldsymbol{\varphi}_n},\quad \Gamma_{zn}=\frac{\boldsymbol{\varphi}_n^T\mathbf{M}\boldsymbol{\iota}_z}{\boldsymbol{\varphi}_n^T\mathbf{M}\boldsymbol{\varphi}_n} \tag{2}$$

When the force $-\left(\Gamma_{xn}\ddot{u}_{gx}+\Gamma_{zn}\ddot{u}_{gz}\right)\mathbf{M}\boldsymbol{\varphi}_n$ is applied to the building only, the equation of motion (Eq. 1) becomes:

$$\mathbf{M}\ddot{\mathbf{u}}_n+\mathbf{C}\dot{\mathbf{u}}_n+\mathbf{K}\mathbf{u}_n=-\left(\Gamma_{xn}\ddot{u}_{gx}+\Gamma_{zn}\ddot{u}_{gz}\right)\mathbf{M}\boldsymbol{\varphi}_n=-\left(\Gamma_{xn}\ddot{u}_{gx}+\Gamma_{zn}\ddot{u}_{gz}\right)\mathbf{s}_n,\quad n=1\sim 3N \tag{3}$$

where $\mathbf{s}_n=\mathbf{M}\boldsymbol{\varphi}_n$ is the modal inertia force of the nth vibration mode; $\mathbf{u}_n$ is the nth modal displacement response and $\mathbf{u}=\sum_{n=1}^{3N}\mathbf{u}_n=\sum_{n=1}^{3N}\boldsymbol{\varphi}_nD_n$. D_n is the generalized modal coordinate, which is a scalar. Notably, $\mathbf{s}_n$ and $-\left(\Gamma_{xn}\ddot{u}_{gx}+\Gamma_{zn}\ddot{u}_{gz}\right)$ respectively reflect the distribution of seismic force in the space and the variation of the seismic force along with time. By replacing $\mathbf{u}_n$ of Eq. 3 with $\boldsymbol{\Phi}_n\mathbf{D}_n$, where

$$\boldsymbol{\Phi}_n=\begin{bmatrix}\boldsymbol{\varphi}_{xn}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\boldsymbol{\varphi}_{zn}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\boldsymbol{\varphi}_{\theta n}\end{bmatrix}_{3N\times3},\quad \mathbf{D}_n=\begin{bmatrix}D_{xn}\\D_{zn}\\D_{\theta n}\end{bmatrix}_{3\times1} \tag{4}$$

, and then multiplying both sides of Eq. 3 with $\boldsymbol{\Phi}_n^T$, the 3DOF modal equation of motion for the nth vibration mode is obtained as:

$$\mathbf{M}_n\ddot{\mathbf{D}}_n+\mathbf{C}_n\dot{\mathbf{D}}_n+\mathbf{K}_n\mathbf{D}_n=-\left(\Gamma_{xn}\ddot{u}_{gx}+\Gamma_{zn}\ddot{u}_{gz}\right)\mathbf{M}_n\mathbf{1},\quad n=1\sim 3N \tag{5}$$

where:

$$\mathbf{M}_n=\begin{bmatrix}\boldsymbol{\varphi}_{xn}^T\mathbf{m}_x\boldsymbol{\varphi}_{xn}&0&0\\0&\boldsymbol{\varphi}_{zn}^T\mathbf{m}_z\boldsymbol{\varphi}_{zn}&0\\0&0&\boldsymbol{\varphi}_{\theta n}^T\mathbf{I}_0\boldsymbol{\varphi}_{\theta n}\end{bmatrix}_{3\times3},\quad \mathbf{1}=\begin{bmatrix}1\\1\\1\end{bmatrix}_{3\times1}$$

$$\mathbf{C}_n=\begin{bmatrix}\boldsymbol{\varphi}_{xn}^T\mathbf{c}_{xx}\boldsymbol{\varphi}_{xn}&\boldsymbol{\varphi}_{xn}^T\mathbf{c}_{xz}\boldsymbol{\varphi}_{zn}&\boldsymbol{\varphi}_{xn}^T\mathbf{c}_{x\theta}\boldsymbol{\varphi}_{\theta n}\\\boldsymbol{\varphi}_{zn}^T\mathbf{c}_{zx}\boldsymbol{\varphi}_{xn}&\boldsymbol{\varphi}_{zn}^T\mathbf{c}_{zz}\boldsymbol{\varphi}_{zn}&\boldsymbol{\varphi}_{zn}^T\mathbf{c}_{z\theta}\boldsymbol{\varphi}_{\theta n}\\\boldsymbol{\varphi}_{\theta n}^T\mathbf{c}_{\theta x}\boldsymbol{\varphi}_{xn}&\boldsymbol{\varphi}_{\theta n}^T\mathbf{c}_{\theta z}\boldsymbol{\varphi}_{zn}&\boldsymbol{\varphi}_{\theta n}^T\mathbf{c}_{\theta\theta}\boldsymbol{\varphi}_{\theta n}\end{bmatrix}_{3\times3},\quad \mathbf{K}_n=\begin{bmatrix}\boldsymbol{\varphi}_{xn}^T\mathbf{k}_{xx}\boldsymbol{\varphi}_{xn}&\boldsymbol{\varphi}_{xn}^T\mathbf{k}_{xz}\boldsymbol{\varphi}_{zn}&\boldsymbol{\varphi}_{xn}^T\mathbf{k}_{x\theta}\boldsymbol{\varphi}_{\theta n}\\\boldsymbol{\varphi}_{zn}^T\mathbf{k}_{zx}\boldsymbol{\varphi}_{xn}&\boldsymbol{\varphi}_{zn}^T\mathbf{k}_{zz}\boldsymbol{\varphi}_{zn}&\boldsymbol{\varphi}_{zn}^T\mathbf{k}_{z\theta}\boldsymbol{\varphi}_{\theta n}\\\boldsymbol{\varphi}_{\theta n}^T\mathbf{k}_{\theta x}\boldsymbol{\varphi}_{xn}&\boldsymbol{\varphi}_{\theta n}^T\mathbf{k}_{\theta z}\boldsymbol{\varphi}_{zn}&\boldsymbol{\varphi}_{\theta n}^T\mathbf{k}_{\theta\theta}\boldsymbol{\varphi}_{\theta n}\end{bmatrix}_{3\times3} \tag{6}$$

D_{xn}, D_{zn}, and $D_{\theta n}$, are referred to as the x- and z-directional modal translations and modal rotation, respectively. $\mathbf{D}_n$ is a vector augmented from the generalized modal coordinate D_n. $\mathbf{D}_n$ reserves the possibility that D_{xn}, D_{zn}, and $D_{\theta n}$ are not equal to each other. Moreover, $\mathbf{M}_n$, $\mathbf{C}_n$, and $\mathbf{K}_n$ reveal the coupled and uncoupled modal properties in the three directions. It is worth noting that the summation of the nine elements of each of the matrices $\mathbf{M}_n$, $\mathbf{C}_n$, and $\mathbf{K}_n$ is equal to the mass, damping, and stiffness

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of the nth SDOF modal equation of motion, respectively. In other words, the mass, damping, and stiffness of the nth SDOF modal equation of motion are expanded into $\mathbf{M}_n$, $\mathbf{C}_n$, and $\mathbf{K}_n$, respectively.

A 3DOF modal stick (Fig. 1a) is constructed in order to represent the 3DOF modal equation of motion (Eq. 5). The degrees of freedom of the 3DOF modal stick (denoted as $\tilde{D}_{xn}$, $\tilde{D}_{zn}$, $\tilde{D}_{\theta n}$) are defined at the lumped mass, which has the x and z-translational masses (denoted as m_{xn}, m_{zn}), and mass moment of inertia (denoted as I_n). The lumped mass is at the end of a rigid beam, whose projected lengths on the X-Y and Z-Y planes are e_{xn} and e_{zn}, respectively. The beam is connected to a rigid column through a rotational spring with stiffness $k_{\theta n}$. The column, which has a unit length, is connected to the ground with two rotational springs with stiffness k_{xn} and k_{zn}. There is an angle β_n between the two rotational springs with stiffness k_{xn} and k_{zn}. For such a simple structure shown as Fig. 1a, its displacement vector, mass matrix, and stiffness matrix are expresses as

$$\tilde{\mathbf{D}} = \begin{bmatrix} \tilde{D}_{xn} \\ \tilde{D}_{zn} \\ \tilde{D}_{\theta n} \end{bmatrix}_{3\times 1}, \quad \tilde{\mathbf{M}} = \begin{bmatrix} m_{xn} & 0 & 0 \\ 0 & m_{zn} & 0 \\ 0 & 0 & I_n \end{bmatrix}_{3\times 3}, \quad C = \cos\beta_n, \quad S = \sin\beta_n$$

$$\tilde{\mathbf{K}} = \begin{bmatrix} k_{xn}C^2 & & symm. \\ k_{xn}SC & k_{zn} + k_{xn}S^2 & \\ -e_{zn}k_{xn}C^2 + e_{xn}k_{xn}SC & -e_{zn}k_{xn}SC + e_{xn}\left(k_{zn} + k_{xn}S^2\right) & k_{\theta n} + e_{xn}^2 k_{zn} + \left(e_{xn}S - e_{zn}C\right)^2 k_{xn} \end{bmatrix}_{3\times 3} \tag{7}$$

By letting $\tilde{\mathbf{D}} = \mathbf{D}_n$, $\tilde{\mathbf{M}} = \mathbf{M}_n$, $\tilde{\mathbf{K}} = \mathbf{K}_n$, the elastic properties of the nth 3DOF modal stick are determined as follows (Eq. 8):

$$\begin{gathered} \beta_n = \tan^{-1}\left(\frac{\boldsymbol{\varphi}_{zn}^T \mathbf{k}_{zx} \boldsymbol{\varphi}_{xn}}{\boldsymbol{\varphi}_{xn}^T \mathbf{k}_{xx} \boldsymbol{\varphi}_{xn}}\right), \quad k_{xn} = \frac{\boldsymbol{\varphi}_{xn}^T \mathbf{k}_{xx} \boldsymbol{\varphi}_{xn}}{C^2}, \quad k_{zn} = \boldsymbol{\varphi}_{zn}^T \mathbf{k}_{zz} \boldsymbol{\varphi}_{zn} - k_{xn} S^2 \\ \begin{bmatrix} e_{xn} \\ e_{zn} \end{bmatrix} = \begin{bmatrix} k_{xn}SC & -k_{xn}C^2 \\ k_{zn} + k_{xn}S^2 & -k_{xn}SC \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{\varphi}_{xn}^T \mathbf{k}_{x\theta} \boldsymbol{\varphi}_{\theta n} \\ \boldsymbol{\varphi}_{zn}^T \mathbf{k}_{z\theta} \boldsymbol{\varphi}_{\theta n} \end{bmatrix}, \quad k_{\theta n} = \boldsymbol{\varphi}_{\theta n}^T \mathbf{k}_{\theta\theta} \boldsymbol{\varphi}_{\theta n} - e_{xn}^2 k_{zn} - \left(e_{xn}S - e_{zn}C\right)^2 k_{xn} \\ m_{xn} = \boldsymbol{\varphi}_{xn}^T \mathbf{m}_x \boldsymbol{\varphi}_{xn}, \quad m_{zn} = \boldsymbol{\varphi}_{zn}^T \mathbf{m}_z \boldsymbol{\varphi}_{zn}, \quad I_n = \boldsymbol{\varphi}_{\theta n}^T \mathbf{I}_0 \boldsymbol{\varphi}_{\theta n} \end{gathered} \tag{8}$$

β_n is approximately equal to zero (i.e. $\mathbf{k}_{zx}$ is neglected) for buildings with all lateral force resisting elements placed along either the X- or Z-axes.

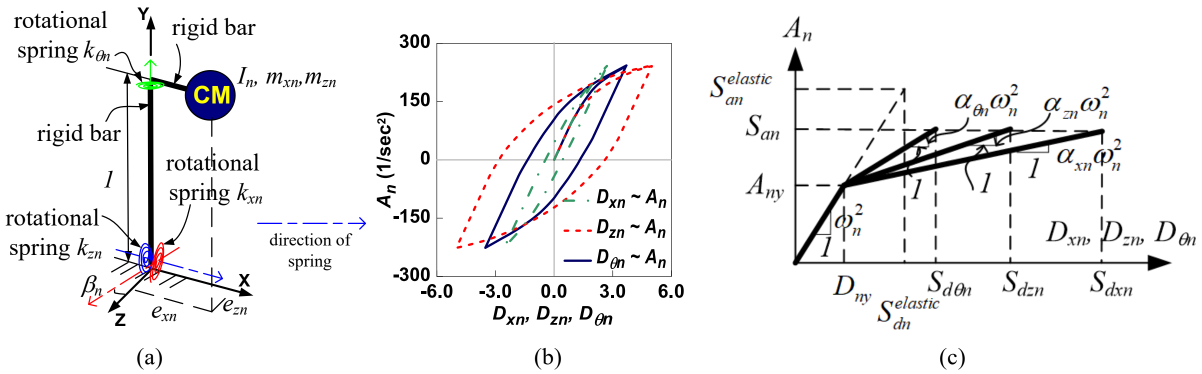

Figure 1. (a) A 3DOF modal stick, (b) three force-displacement curves, (c) three bilinear force-displacement curves.

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While applying a modal inertia force vector to a symmetrical building, there is only one force-displacement curve, which represents the roof displacement versus base shear relationship. However, while applying a modal inertia force vector to an asymmetrical building, there are three force-displacement curves, which represents two roof displacement-base shear relationships and one roof rotation-base torque relationship (Fig. 1b). The SDOF modal system is capable of describing the one force-displacement relationship. However, the SDOF modal system is obviously insufficient to describe the three force-displacement relationships simultaneously. By converting the three force-displacement curves into the acceleration-displacement-response-spectra (ADRS) format, the three curves overlap when the building is elastic. In addition, the three force-displacement curves bifurcate when the building becomes inelastic (Fig. 1b). This bifurcation indicates that the modal responses in the three directions are different from each other (i.e., non-proportional) when asymmetrical buildings are in an inelastic state. By bi-linearizing the three force-displacement curves, the post-yielding stiffness ratios α_{xn}, α_{znn}, $\alpha_{\theta n}$, as well as the yielding accelerations A_{yxn}, A_{yzn}, $A_{y\theta n}$ are obtained (Fig. 1c). Using these parameter values, the inelastic properties of the 3DOF modal system, including the yielding moments (denoted as M_{yxn}, M_{yzn}, $M_{y\theta n}$) and inelastic stiffness of the three rotational springs (denoted as k'_{xn}, k'_{zn}, $k'_{\theta n}$) are determined and shown as Eq. 9.

$$M_{yxn} = A_{xny} m_{xn}, \quad M_{yzn} = A_{zny} m_{zn} \tag{9a}$$

$$M_{y\theta n} = A_{\theta ny} I_n + A_{xny} m_{xn} e_{zn} - A_{zny} m_{zn} e_{xn} \tag{9b}$$

$$k'_{xn} = \frac{m_{xn}}{\dfrac{\dfrac{m_{xn}}{k_{xn}} + \dfrac{\left(I_n + m_{xn}e_{zn} - m_{zn}e_{xn}\right)e_{zn}}{k_{\theta n}}}{\alpha_{xn}} - \dfrac{\left(I_n + m_{xn}e_{zn} - m_{zn}e_{xn}\right)e_{zn}}{k'_{\theta n}}} \tag{9c}$$

$$k'_{zn} = \frac{m_{zn}}{\dfrac{\dfrac{m_{zn}}{k_{zn}} - \dfrac{\left(I_n + m_{xn}e_{zn} - m_{zn}e_{xn}\right)e_{xn}}{k_{\theta n}}}{\alpha_{zn}} + \dfrac{\left(I_n + m_{xn}e_{zn} - m_{zn}e_{xn}\right)e_{xn}}{k'_{\theta n}}} \tag{9d}$$

$$k'_{\theta n} = k_{\theta n} \cdot \alpha_{\theta n} \tag{9e}$$

Up to here, all the elastic and inelastic properties of a 3DOF modal stick are determined (Eqs. 8 and 9). By solving (using the step-by-step integration as an example) the 3DOF modal equation of motion (Eq. 5), the nth modal response history $\mathbf{D}_n$ can be obtained. The total displacement response $\mathbf{u}$ of the original building is then calculated as:

$$\mathbf{u} = \sum_{n=1}^{3N} \mathbf{u}_n = \sum_{n=1}^{3N} \mathbf{\Phi}_n \mathbf{D}_n = \sum_{n=1}^{3N} \begin{bmatrix} \boldsymbol{\varphi}_{xn} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\varphi}_{zn} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\varphi}_{\theta n} \end{bmatrix}_{3N\times 3} \begin{bmatrix} D_{xn} \\ D_{zn} \\ D_{\theta n} \end{bmatrix}_{3\times 1} \tag{10}$$

Certainly, there are three vibration modes for each 3DOF modal system. In order to avoid confusion with the vibration modes of the original asymmetrical building, these three modes of a 3DOF modal system are called ‘sub-modes’. It has been mathematically demonstrated that one of the sub-modes is active with modal participation factor equal to one. The other two sub-modes are spurious as modal participation factors are equal to zero, i.e. the spurious sub-modes do not contribute to the vibration of an elastic 3DOF modal system. Moreover, the vibration period of the active sub-mode is equal to the nth vibration period of the original asymmetrical building. The mode shape of the active

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sub-mode is equal to $[1\ 1\ 1]^T$, i.e. $D_{xn} = D_{zn} = D_{\theta n}$ for elastic 3DOF modal systems. Therefore, the 3DOF modal system is identical to the conventional SDOF modal system in an elastic state. In other words, the 3DOF modal system, a result of the augmented modal dimensions, completely complies with the basic principle of structural dynamics. Nevertheless, the non-proportionality between modal translations (D_{xn} and D_{zn}) and modal rotation ($D_{\theta n}$) of an inelastic asymmetrical building can be captured by using the proposed 3DOF modal systems.

2.2. Non-proportional damping reflected at a modal level

When an N-story asymmetrical building is proportionally damped (i.e., $\mathbf{C}=\alpha\mathbf{M}+\beta\mathbf{K}$), its corresponding 3DOF modal stick is also proportionally damped, i.e.,

$$\mathbf{C}_n=\begin{bmatrix}\boldsymbol{\varphi}_{xn} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\varphi}_{zn} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \boldsymbol{\varphi}_{\theta n}\end{bmatrix}^T \mathbf{C}\begin{bmatrix}\boldsymbol{\varphi}_{xn} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\varphi}_{zn} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \boldsymbol{\varphi}_{\theta n}\end{bmatrix}=\begin{bmatrix}\boldsymbol{\varphi}_{xn} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\varphi}_{zn} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \boldsymbol{\varphi}_{\theta n}\end{bmatrix}^T \left(\alpha\mathbf{M}+\beta\mathbf{K}\right)\begin{bmatrix}\boldsymbol{\varphi}_{xn} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\varphi}_{zn} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \boldsymbol{\varphi}_{\theta n}\end{bmatrix}$$
$$=\alpha\mathbf{M}_n+\beta\mathbf{K}_n \tag{11}$$

Therefore, when an N-story asymmetrical building is non-proportionally damped (i.e., $\mathbf{C}\neq\alpha\mathbf{M}+\beta\mathbf{K}$), its corresponding 3DOF modal stick is also non-proportionally damped, i.e., $\mathbf{C}_n\neq\alpha\mathbf{M}_n+\beta\mathbf{K}_n$. That is to say, the damping characteristic of an N-story building is retained at a modal level. Hence, the augmented modal dimensions allow the modal responses in the three directions to be different from each other when asymmetrical buildings are non-proportionally damped. This feature benefits the modal response history analyses of asymmetrical buildings with supplemental damping or with soil-structure interaction [6].

Lin and Tsai [5] investigated a three-story building with linear viscous dampers placed in both the x and z-directions. The damping coefficients were assigned so that the damping ratio in each direction is 30%. Figure 2a-c shows the three-directional modal responses (i.e., $\tilde{D}_{xn}$, $\tilde{D}_{zn}$, $\tilde{D}_{\theta n}$) of the 1st, 2nd, and 3rd vibration modes of the building, respectively, subjected to a scaled bi-directional El Centro earthquake. It indicates that the three-directional modal responses of each vibration mode are obviously different from each other because of the supplemental damping effect. Figure 2d-f shows the response histories of the 3rd, 2nd, and 1st stories of the building, respectively. In Fig. 2d-f, the abbreviation 3MA stands for the seismic responses obtained from using the 3DOF modal response history analysis. Additionally, the abbreviation SMA stands for the seismic responses obtained from the response history analysis of the complete finite element model of the building. Figure 2d-f indicates that the modal response history analysis with 3DOF modal systems effectively estimated the floor displacement histories of the non-proportionally damped building. The excellent feature of augmented modal dimensions for seismic evaluation of non-proportionally damped buildings is verified.

3. Conclusions

The augmented modal dimensions explicitly reveal the translational and rotational modal properties of each vibration mode of asymmetrical buildings. In addition, the augmented modal dimensions is capable of reflecting the non-proportional modal responses in the three directions when asymmetrical buildings are inelastic or non-proportionally damped. The augmented modal dimensions can be applied not only to asymmetrical buildings but also to vertically irregular buildings. Furthermore, the concept of augmented modal dimensions benefits the development of novel tuned mass dampers. It is worth putting more effort on the study of augmented modal dimensions in the future.

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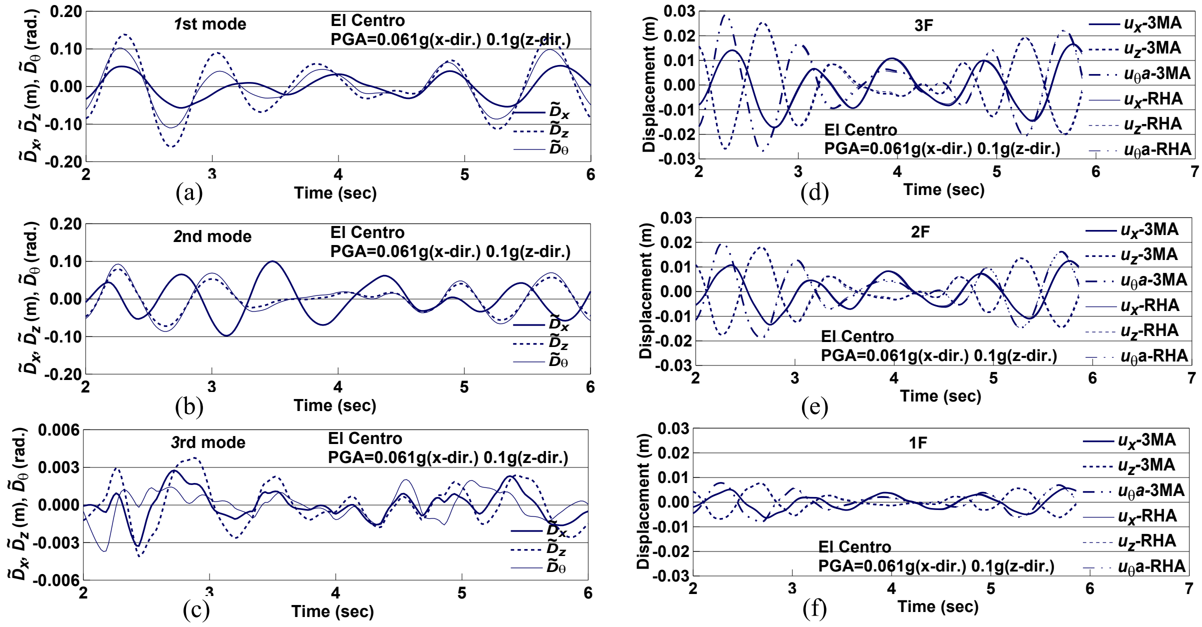

Figure 2. The three-directional modal responses of the (a) 1st, (b) 2nd, and (c) 3rd vibration modes of a non-proportionally damped three-story building. The response histories of the (d) 3rd, (e) 2nd, and (f) 1st floors of the three-story building.

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